ON THE APPLICATION OF THE RODRIGUES-HAMILTON AND CAYLEY-KLEIN PARAMETERS IN THE APPLIED THEORY OF GYROSCOPES

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In the present paper the analytic techniques based on the use of the Rodrigues-Hamilton and Cayley-Klein parameters are brought to bear on the applied (precessional) theory of gyroscopes. The kinematics of a single-rotor gyropendulum [1] and of a Geckeler-Ishlinskii horizontal gyrocompass [2] are considered, as also are the dynamic equations of the precessional motion of a gyropendulum within finite angles.

1. The motion of a single-rotor gyropendulum will be considered relative to a trihedron $Ox^\circ y^\circ z^\circ$ with right-handed coordinates and with origin at the center of gimbals, oriented along the velocity vector of the point of suspension [1 and 2].

We shall also introduce the Résal trihedron $Oxyz$ with origin at the same point O, which does not participate in the neutral rotation of the gyroscope.

The transfer from the trihedron $Ox^\circ y^\circ z^\circ$ to the trihedron $Oxyz$ is effected by means of two successive finite rotations through the angles α and β of the outer and inner rings, respectively*.

The direction cosines between the axes of the trihedra which we have introduced, form the matrix $a = \| a_{jk} \|$ $(i, k = 1, 2, 3)$, defined by the Table:

	x°	y°	z°
x	$a_{11} = \cos\beta$	$a_{12} = \sin\alpha\sin\beta$	$a_{13} = -\cos\alpha\sin\beta$
y	$a_{21} = 0$	$a_{22} = \cos\alpha$	$a_{23} = \sin\alpha$
z	$a_{31} = \sin\beta$	$a_{32} = -\sin\alpha\cos\beta$	$a_{33} = \cos\alpha\cos\beta$

Let λ_s $(s = 0, 1, 2, 3)$ be the Rodrigues-Hamilton parameters relative to the vector of finite rotation θ which takes the system $x^\circ y^\circ z^\circ$ into xyz.

The expressions for the direction cosines a_{jk} in terms of the parameters λ_s will have the form

*) See Fig.9 of [1]. There these angles were denoted, respectively,by θ and ψ.

$$a_{11} = \lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2, \quad a_{12} = 2(\lambda_0\lambda_3 + \lambda_1\lambda_2), \quad a_{13} = 2(\lambda_1\lambda_3 - \lambda_0\lambda_2) \quad (1.1)$$
$$a_{21} = 2(\lambda_1\lambda_2 - \lambda_0\lambda_3), \quad a_{22} = \lambda_0^2 + \lambda_2^2 - \lambda_3^2 - \lambda_1^2, \quad a_{23} = 2(\lambda_0\lambda_1 + \lambda_2\lambda_3)$$
$$a_{31} = 2(\lambda_0\lambda_2 + \lambda_3\lambda_1), \quad a_{32} = 2(\lambda_2\lambda_3 - \lambda_0\lambda_1), \quad a_{33} = \lambda_0^2 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2$$

The connection between the angles α and β and the parameters λ are established by Formulas [3]

$$\lambda_0 = \cos {}^1/_2\,\alpha \cos {}^1/_2\,\beta, \quad \lambda_1 = \sin {}^1/_2\,\alpha \cos {}^1/_2\,\beta, \quad \lambda_2 = \cos {}^1/_2\,\alpha \sin {}^1/_2\,\beta, \quad \lambda_3 = \sin {}^1/_2\,\alpha \sin {}^1/_2\,\beta \quad (1.2)$$

The relations

$$\sin\alpha = 2(\lambda_0\lambda_1 + \lambda_2\lambda_3), \qquad \sin\beta = 2(\lambda_0\lambda_2 + \lambda_3\lambda_1) \quad (1.3)$$

follow from (1.2) also.

2. Let p, q, r be the projections of the angular velocity of the trihedron xyz connected to the inner ring (casing) onto its own axes. We have

$$p = \alpha^{\cdot} a_{11} + \frac{v}{R} a_{12} + r^\circ a_{13}, \qquad q = \beta^{\cdot} + \frac{v}{R} a_{22} + r^\circ a_{23}, \qquad r = \alpha^{\cdot} a_{31} + \frac{v}{R} a_{32} + r^\circ a_{33} \quad (2.1)$$

Here v is the velocity of the point of suspension, r° is the projection onto the z°-axis of the angular velocity of the trihedron $x^\circ y^\circ z^\circ$.

The expressions for p, q, r in terms of the parameters λ_s and their derivatives have the form [3]

$$p = 2[\lambda_0\lambda_1^{\cdot} - \lambda_1\lambda_0^{\cdot} + \lambda_3\lambda_2^{\cdot} - \lambda_2\lambda_3^{\cdot} + q^\circ(\lambda_0\lambda_3 + \lambda_1\lambda_2) + r^\circ(\lambda_1\lambda_3 - \lambda_0\lambda_2)] \qquad (q^\circ = v/R)$$
$$q = 2[\lambda_0\lambda_2^{\cdot} - \lambda_2\lambda_0^{\cdot} + \lambda_1\lambda_3^{\cdot} - \lambda_3\lambda_1^{\cdot} + {}^1/_2\, q^\circ(\lambda_0^2 + \lambda_2^2 - \lambda_3^2 - \lambda_1^2) + r^\circ(\lambda_0\lambda_1 + \lambda_2\lambda_3)] \quad (2.2)$$
$$r = 2[\lambda_0\lambda_3^{\cdot} - \lambda_3\lambda_0^{\cdot} + \lambda_2\lambda_1^{\cdot} - \lambda_1\lambda_2^{\cdot} + q^\circ(\lambda_2\lambda_3 - \lambda_0\lambda_1) + {}^1/_2\, r^\circ(\lambda_0^2 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2)]$$

The parameters λ_s are related by Equation

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 \quad (2.3)$$

from which follows the differential relation

$$\lambda_0\lambda_0^{\cdot} + \lambda_1\lambda_1^{\cdot} + \lambda_2\lambda_2^{\cdot} + \lambda_3\lambda_3^{\cdot} = 0 \quad (2.4)$$

From (2.2) and (2.4) it is easy to obtain the Rodrigues-Hamilton equations for the derivatives of λ_s, generalized to the case of a moving base. The latter form a system of linear differential equations having the form

$$2\lambda_0^{\cdot} = -p\lambda_1 - (q - q^\circ)\lambda_2 - (r - r^\circ)\lambda_3, \quad 2\lambda_1^{\cdot} = p\lambda_0 + (r + r^\circ)\lambda_2 - (q + q^\circ)\lambda_3 \quad (2.5)$$
$$2\lambda_2^{\cdot} = (q - q^\circ)\lambda_0 + p\lambda_3 - (r + r^\circ)\lambda_1, \quad 2\lambda_3^{\cdot} = (r - r^\circ)\lambda_0 + (q + q^\circ)\lambda_1 - p\lambda_2$$

The classical Rodrigues-Hamilton equations (for a fixed base) are obtained from (2.5) by letting $q^\circ = r^\circ = 0$ in them.

3. Let us consider here the kinematics of the motion of the sensing element (gyrosphere) of a two-rotor horizontal gyrocompass [2].

Let $Ox^\circ y^\circ z^\circ$ be a trihedron with right-handed coordinates, oriented, as in the gyropendulum case, along the velocity vector of the point of suspension.

Let us also introduce a moving trihedron $Oxyz$ connected to the gyrosphere and obtained from $Ox^\circ y^\circ z^\circ$ by a sequence of three finite rotations through the angles α, β, γ.

In this case the expressions for the direction cosines will have the form

$$a_{11} = \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma, \quad a_{12} = \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma$$
$$a_{13} = -\cos\beta\cos\gamma, \quad a_{21} = -\sin\alpha\cos\beta, \quad a_{22} = \cos\alpha\cos\beta, \quad a_{23} = \sin\beta \quad (3.1)$$
$$a_{31} = \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma, \quad a_{32} = \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma, \quad a_{33} = \cos\beta\cos\gamma$$

Comparing (1.1) (these formulas, of course, retain their own structure) with (3.1), we can convince ourselves of the validity of Formulas (3.2)

$$\tan\alpha = \frac{2(\lambda_0\lambda_3 - \lambda_1\lambda_2)}{\lambda_0^2 + \lambda_2^2 - \lambda_3^2 - \lambda_1^2}, \quad \sin\beta = 2(\lambda_0\lambda_1 + \lambda_2\lambda_3), \quad \tan\gamma = \frac{2(\lambda_0\lambda_2 - \lambda_1\lambda_3)}{\lambda_0^2 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2}$$

It is somewhat complicated to establish the exxpressions for the Rodrigues-Hamilton parameters in terms of functions of the angles α, β, γ. The transfer from the system $x^\circ y^\circ z^\circ$ to the system xyz is determined by the three successive finite rotations

$$\theta_1 = 2e_1 \tan {}^1/_2\,\alpha, \qquad \theta_2 = 2e_2 \tan {}^1/_2\,\beta, \qquad \theta_3 = 2e_3 \tan {}^1/_2\,\gamma \tag{3.3}$$

Here, the unit vector e_s gives the direction of the axes around which the sequence of rotations through the angles α, β, γ are accomplished.

In order to obtain expressions for λ_s in terms of functions of the angles α, β, γ, we can apply to (3.3) twice the procedure of adding on finite rotations. However, it is simpler to make use of the Lur'e theorem on the transposition of finite rotations. According to this theorem the sequence of rotations θ_1 and θ_2 is equivalent to the series

$$\theta_1' = \theta_2 = 2e_2 \tan {}^1/_2\,\beta, \qquad \theta_2' = 2e_1' \tan {}^1/_2\,\alpha \tag{3.4}$$

Here e_1' is the unit vector to which e_1 transfers under rotation θ_2.

Therefore, denoting by λ_s' the Rodrigues parameters corresponding to the resultant rotation θ_{12} and by ν_s and μ_s ($s = 0, 1, 2, 3$) the parameters of the component rotations, according to the theorem on transposed rotations we set

$$\begin{aligned} &\nu_0 = \cos {}^1/_2\,\beta, \quad \nu_1 = \sin {}^1/_2\,\beta, \quad \nu_2 = \nu_3 = 0 \\ &\mu_0 = \cos {}^1/_2\,\alpha, \quad \mu_1 = \mu_2 = 0, \quad \mu_3 = \sin {}^1/_2\,\alpha \end{aligned} \tag{3.5}$$

For λ_s' we have Formulas

$$\lambda_0' = \nu_0\mu_0 - \sum_{s=1}^{3} \nu_s\mu_s, \qquad \lambda_s' = \nu_s\mu_0 + \mu_s\nu_0 + \sum_{r=1}^{3}\sum_{t=1}^{3} \in_{rts}\mu_r\nu_t \tag{3.6}$$

Here $\in_{rts}$ is the Levi-Civita symbol [3]. Hence it follows that

$$\begin{aligned} &\lambda_0' = \cos {}^1/_2\,\alpha \cos {}^1/_2\,\beta, \qquad \lambda_1' = \cos {}^1/_2\,\alpha \sin {}^1/_2\,\beta \\ &\lambda_2' = \sin {}^1/_2\,\alpha \sin {}^1/_2\,\beta \qquad \lambda_3' = \sin {}^1/_2\,\alpha \cos {}^1/_2\,\beta \end{aligned} \tag{3.7}$$

Further, we should consider the sequence of rotations θ_{12} and θ_3 by again making use of the theorem of transposed rotations.

In this case we should right away set

$$\nu_0 = \cos {}^1/_2\gamma, \quad \nu_1 = 0, \quad \nu_2 = \sin {}^1/_2\gamma, \quad \nu_3 = 0, \quad \mu_s = \lambda_s \tag{3.8}$$

Denoting the desired Rodrigues-Hamilton parameters by λ_s and again using the formulas of structure (3.6), we obtain

$$\begin{aligned} \lambda_0 &= \cos {}^1/_2\,\alpha \cos {}^1/_2\,\beta \cos {}^1/_2\,\gamma - \sin {}^1/_2\,\alpha \sin {}^1/_2\,\beta \sin {}^1/_2\,\gamma \\ \lambda_1 &= \cos {}^1/_2\,\alpha \sin {}^1/_2\,\beta \cos {}^1/_2\,\gamma - \sin {}^1/_2\,\alpha \cos {}^1/_2\,\beta \sin {}^1/_2\,\gamma \\ \lambda_2 &= \cos {}^1/_2\,\alpha \cos {}^1/_2\,\beta \sin {}^1/_2\,\gamma + \sin {}^1/_2\,\alpha \sin {}^1/_2\,\beta \cos {}^1/_2\,\gamma \\ \lambda_3 &= \sin {}^1/_2\,\alpha \cos {}^1/_2\,\beta \cos {}^1/_2\,\gamma + \cos {}^1/_2\,\alpha \sin {}^1/_2\,\beta \sin {}^1/_2\,\gamma \end{aligned} \tag{3.9}$$

The generalized Rodrigues-Hamilton equations for the horizontal gyrocompass may be obtained from (2.5) by setting $p = 0$ in them [2]. We have

$$\begin{aligned} &2\lambda_0^{\cdot} = -(q - q^\circ)\lambda_2 - (r - r^\circ)\lambda_3, \qquad 2\lambda_1^{\cdot} = (r + r^\circ)\lambda_2 - (q + q^0)\lambda_3 \\ &2\lambda_2^{\cdot} = (q - q^\circ)\lambda_0 - (r + r^0)\lambda_1, \qquad 2\lambda_3^{\cdot} = (r - r^\circ)\lambda_0 + (q + q^\circ)\lambda_1 \end{aligned} \tag{3.10}$$

Here

$$q = q^\circ a_{22} + (r^\circ + \alpha^{\cdot})\, a_{23} + \gamma^{\cdot}, \qquad r = q^\circ a_{32} + (r^\circ + \alpha^{\cdot})\, a_{33} + \beta^{\cdot} \sin\gamma \tag{3.11}$$

and, further, the a_{jk} are expressed by Formulas (3.1).

4. The application of the Rodrigues-Hamilton parameters to the dynamic problems of the applied theory of gyroscopes, of course, meets with great

difficulties in comparison with the purely kinematic aspects presented in Sections 1 to 3.

However, here too it is possible in certain cases to simplify the analytic aspects of the problem by means of a transfer from the Euler-Résal angles to the R drigues-Hamilton parameters, by which we can arrive at linear differential equations without a preliminary linearization of the original equations.

As an example which will clarify this concept, let us consider the equations of precessional motion of a single-rotor gyropendulum set on a fixed base. Relative to the Résal axes these equations have the form [1]

$$Hq = M_x, \qquad -Hp = M_y \tag{4.1}$$

where, taking (2.1) into account, we should assume

$$p = \alpha^{\cdot} \cos\beta, \quad q = \beta^{\cdot}, \quad M_x = -Pl\sin\alpha, \quad M_y = -Pl\cos\alpha\sin\beta \tag{4.2}$$

Here, as before, α and β are the angles of rotation of the outer and inner rings, Pl is the pendulum moment, H is the natural kinetic moment of the gyroscope, which we shall take to be constant. It is well-known [1] that the structure of Equations (4.1) is preserved also for an arbitrary trihedron $Ox^*y^*z^*$ with origin at the center of the gimbals, where the z^*-axis coincides with the z-axis of the thrihedron $Oxyz$.

Let us introduce the trihedron $Ox^*y^*z^*$ connected to the rotor, formed by a rotation of the trihedron $Oxyz$ around the z-axis through an angle γ . The position of the rotor will thus be determined by the angles α , β , γ .

The Rodrigues-Hamilton parameters corresponding to the sequence of three rotations through the angles α , β , γ , transferring the system $Ox^\circ y^\circ z^\circ$ to $Ox^*y^*z^*$, are determined by Formulas

$$\begin{aligned}
\lambda_0 &= \cos {}^1/_2\,\gamma \cos {}^1/_2\,\alpha \cos {}^1/_2\,\beta - \sin {}^1/_2\,\gamma \sin {}^1/_2\,\alpha \sin {}^1/_2\,\beta \\
\lambda_1 &= \cos {}^1/_2\,\gamma \sin {}^1/_2\,\alpha \cos {}^1/_2\,\beta + \sin {}^1/_2\,\gamma \cos {}^1/_2\,\alpha \sin {}^1/_2\,\beta \\
\lambda_2 &= \cos {}^1/_2\,\gamma \cos {}^1/_2\,\alpha \sin {}^1/_2\,\beta - \sin {}^1/_2\,\gamma \sin {}^1/_2\,\alpha \cos {}^1/_2\,\beta \\
\lambda_3 &= \sin {}^1/_2\,\gamma \cos {}^1/_2\,\alpha \cos {}^1/_2\,\beta + \cos {}^1/_2\,\gamma \sin {}^1/_2\,\alpha \sin {}^1/_2\,\beta
\end{aligned} \tag{4.3}$$

obtained analogously as in Section 3 for the horizontal gyrocompass.

The equations of precessional motion relative to the axes $x^*y^*z^*$ have the form

$$Hq^* = M_{x^*}, \qquad -Hp^* = M_{y^*} \tag{4.4}$$

where we should take

$$\begin{aligned}
p^* &= p\cos\gamma + q\sin\gamma, & q^* &= -p\sin\gamma + q\cos\gamma \\
M_{x^*} &= M_x\cos\gamma + M_y\sin\gamma, & M_{y^*} &= -M_x\sin\gamma + M\cos\gamma
\end{aligned} \tag{4.5}$$

Equations (4.4) in the variables λ, determined from Formulas (4.3) have the form

$$\begin{aligned}
\lambda_0\lambda_1^{\cdot} - \lambda_1\lambda_0^{\cdot} + \lambda_3^{\cdot}\lambda_2^{\cdot} - \lambda_2\lambda_3^{\cdot} + \omega(\lambda_1\lambda_3 - \lambda_0\lambda_2) &= 0 \\
\lambda_0\lambda_2^{\cdot} - \lambda_2\lambda_0^{\cdot} + \lambda_1\lambda_3^{\cdot} - \lambda_3\lambda_1^{\cdot} + \omega(\lambda_0\lambda_1 + \lambda_2\lambda_3) &= 0
\end{aligned} \qquad \left(\omega = \frac{Pl}{H}\right) \tag{4.6}$$

Equations (2.4) should be combined with Equations (4.6). The fourth equation can be obtained with the help of the integral of the energy, which in the given case has the form [5]

$$\frac{H^2}{2C} + Pl\cos\alpha\cos\beta = h_1 \tag{4.7}$$

where C is the polar moment of inertia of the rotor, while the quantity

$$H \equiv Cr^* = C(\gamma^{\cdot} + \alpha^{\cdot}\sin\beta) = h_2 \tag{4.8}$$

determines the cyclic integral in the steady-state rotation of the gyroscope.

From (4.7) and (4.8) follows the validity of the expression

$$r^* + \frac{Pl}{H}\cos\alpha\cos\beta = h \qquad (h = \text{const}) \tag{4.9}$$

In the variables λ, Equation (4.9) will be

$$\lambda_0\lambda_3^{\cdot} - \lambda_3\lambda_0^{\cdot} + \lambda_1^{\cdot}\lambda_2 - \lambda_3^{\cdot}\lambda_1 + {}^1/_2\omega\,(\lambda_0^2 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2) = {}^1/_2 h \qquad (4.10)$$

Solving Equations (4.6), (2.4) and (4.10) with respect to the derivatives of the Rodrigues-Hamilton parameters, we obtain four linear differential equations with constant coefficients, which can be split up into two independent systems of the form

$$2\lambda_0^{\cdot} = (\omega - h)\,\lambda_3, \qquad 2\lambda_1^{\cdot} = (\omega + h)\,\lambda_2$$
$$2\lambda_3^{\cdot} = -(\omega - h)\,\lambda_0, \qquad 2\lambda_2^{\cdot} = -(\omega + h)\,\lambda_1 \qquad (4.11)$$

Here, by virtue of (4.9), with due regard to the initial conditions we should set

$$\omega - h = \frac{Pl}{H} - \gamma^{\cdot}(0) - \alpha^{\cdot}(0)\sin\beta(0) - \frac{Pl}{H}\cos\alpha(0)\cos\beta(0)$$
$$\omega + h = \frac{Pl}{H} + \gamma^{\cdot}(0) + \alpha^{\cdot}(0)\sin\beta(0) + \frac{Pl}{H}\cos\alpha(0)\cos\beta(0) \qquad (4.12)$$

Equations (4.11) can be still further simplified if we pass to the Cayley-Klein parameters by setting

$$\lambda_0 + i\lambda_3 = u_1, \qquad \lambda_1 + i\lambda_2 = u_2 \qquad (4.13)$$

Relative to the Cayley-Klein parameters u_1 and u_2 we get Equations

$$u_1^{\cdot} = -(\omega - h)\,iu_1, \qquad u_2^{\cdot} = -(\omega + h)\,iu_2 \qquad (i = \sqrt{-1}) \qquad (4.14)$$

Equations (4.11) (or (4.14)) are integrated to give the solution of the problem.

5. In linear cases the equations we have obtained admit of an easy degeneration.

For small angles α and β, as λ_s we can take the expressions (5.1)

$$\lambda_0 = \cos{}^1/_2\gamma, \quad \lambda_1 = {}^1/_2(\alpha\cos{}^1/_2\gamma + \beta\sin{}^1/_2\gamma), \quad \lambda_2 = {}^1/_2(\beta\cos{}^1/_2\gamma - \alpha\sin{}^1/_2\gamma), \quad \lambda_3 = \sin{}^1/_2\gamma$$

Further, from (4.8) by neglecting terms of the second order of smallness, we obtain

$$r = \gamma^{\cdot} = \gamma^{\cdot}(0) = \text{const} \qquad (5.2)$$

Under these conditions the first two of Equations (4.11) become identities; from the other two we obtain equations of the form

$$\alpha^{\cdot}\cos{}^1/_2\gamma + \beta^{\cdot}\sin{}^1/_2\gamma = (Pl/H)(\beta\cos{}^1/_2\gamma - \alpha\sin{}^1/_2\gamma)$$
$$-\alpha^{\cdot}\sin{}^1/_2\gamma + \beta^{\cdot}\cos{}^1/_2\gamma = -(Pl/H)(\alpha\cos{}^1/_2\gamma + \beta\sin{}^1/_2\gamma) \qquad (5.3)$$

whence ensue the equations of small oscillations of a gyroscopic pendulum

$$H\beta^{\cdot} = -Pl\alpha, \qquad H\alpha^{\cdot} = Pl\beta \qquad (5.4)$$

which, of course, are directly obtainable after a linearization of system (4.1).

In conclusion we note that by virtue of the arbitrariness of the choice of the system of coordinates $Ox^*y^*z^*$ with origin at the center of suspension and with the z^*-axis coincident with the z-axis of the Résal trihedron, the motion of the system along the coordinate γ can be chosen so as to satisfy the condition $h = 0$. This circumstance simplifies matters still further.

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